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Weak MPCEP and *CEPMP inverses

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Abstract

We generalize the systems of equations, which introduced the MP-CEP and *CEPMP inverses, using a minimal rank weak Drazin inverse and a minimal rank right weak Drazin inverse. In order to solve new generalized systems of matrix equations, we define new types of generalized inverses, the so-called weak MPCEP and *CEPMP inverses. The DMP, MPD, MPCEP and *CEPMP inverses are particular kinds of weak MPCEP and *CEPMP inverses. We show characterizations and formulae for weak MPCEP and *CEPMP inverses as well as their perturbation results. As application of weak MPCEP and *CEPMP inverses, we prove solvability of certain linear equations and recover the main application of the Moore–Penrose inverse.

1 Introduction

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. Denote by $\mathcal{N}(A)$, $\mathcal{R}(A)$, $rank(A)$ and A^*, respectively, the null space, range, rank and conjugate transpose of $A \in \mathbb{C}^{m\times n}$.

Solving systems of matrix equations, various kinds of generalized inverses are defined. The principal application of generalized inverses is in solving linear systems, where they are used in much the same way as ordinary inverses in the nonsingular case.

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The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is unique $X = A^{\dagger} \in \mathbb{C}^{n\times m}$ satisfying $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$ [2]. If only $XAX = X$ (or $AXA = A$) holds, then X is an outer (or inner) inverse of A. For subspaces T of $\mathbb{C}^n$ and S of $\mathbb{C}^m$ with dimensions $s \leq rank(A)$ and $m - s$, respectively, in the case that $XAX = X$, $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$, $X = A^{(2)}_{T,S}$ is unique (if it exists) [2].

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with the index $k = \operatorname{ind}(A)$ is unique $X = A^D \in \mathbb{C}^{n\times n}$ such that $A^{k+1}X = A^k$, $XAX = X$ and $AX = XA$ [2]. If $\operatorname{ind}(A) = 1$, $A^{\#} = A^D$ is the group inverse of A. The core–EP inverse of $A \in \mathbb{C}^{n\times n}$ is unique $X = A^{\textcircled{\dagger}} \in \mathbb{C}^{n\times n}$ satisfying $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, where $k = \operatorname{ind}(A)$ [16]. Notice that $A^{\textcircled{\dagger}} = A^D A^k (A^k)^{\dagger}$ [6]. As a dual core-EP, the $*$core–EP inverse of A is expressed by $A_{\textcircled{\dagger}} = (A^k)^{\dagger} A^k A^D$ [24]. When $\operatorname{ind}(A) = 1$, $A^{\textcircled{\#}} = A^{\textcircled{\dagger}}$ is the core inverse of A and $A_{\textcircled{\#}} = A_{\textcircled{\dagger}}$ is the dual core inverse of A [1].

Combining in adequate manners the above mentioned generalized inverses, some known generalized inverses can be represented:

- the DMP inverse $A^{D,\dagger} = A^D A A^{\dagger}$ [10],
- the MPD inverse $A^{\dagger,D} = A^{\dagger} A A^D$ [10],
- the CMP inverse $A^{c,\dagger} = A^{\dagger} A A^D A A^{\dagger}$ [11],
- the weak group inverse $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$ [20],
- the m-weak group inverse $A^{\textcircled{w}_m} = (A^{\textcircled{\dagger}})^{m+1} A^m$, where $m \in \mathbb{N}$ [25].

As a compose of the Moore-Penrose inverse and core-EP inverse, the MPCEP inverse was defined in [4] for bounded linear Hilbert space operators. The MPCEP inverse of $A \in \mathbb{C}^{n\times n}$ is uniquely determined solution to the system

$$XAX = X, \qquad XA = A^{\dagger} A A^{\textcircled{\dagger}} A \qquad \text{and} \qquad AX = AA^{\textcircled{\dagger}}, \tag{1}$$

which can be expressed as

$$A^{\dagger,\textcircled{\dagger}} = A^{\dagger} A A^{\textcircled{\dagger}}.$$

The $*$CEPMP inverse of A is unique solution to

$$XAX = X, \qquad AX = AA_{\textcircled{\dagger}} AA^{\dagger} \qquad \text{and} \qquad XA = A_{\textcircled{\dagger}} A, \tag{2}$$

and it is given by

$$A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger}} A A^{\dagger}.$$

Interesting properties of the MPCEP inverse for complex matrices can be found in [15]. The definition of the MPCEP inverse was generalized to operators between two Hilbert spaces in [18] and to quaternion matrices in [7, 8, 9]. New extensions of the MPCEP inverse were proposed in [22, 23].

As an extension of the Drazin inverse, a weak Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with $k = \text{ind}(A)$, was presented in [3] as a solution $X \in \mathbb{C}^{n\times n}$ of the equation $XA^{k+1} = A^k$. Remark that the weak Drazin inverse is not uniquely determined, but it is easier to calculate weak Drazin inverse than Drazin inverse and applied it instead of Drazin inverse in the theory of differential equations, Markov chains and so on. A minimal rank weak Drazin inverse X of A [3] satisfies

$$XA^{k+1} = A^k \quad \text{and} \quad rank(X) = rank(A^D).$$

The Drazin inverse A^D is unique minimal rank weak Drazin inverse of A which commutes with A. Some known generalized inverses, such as the DMP inverse, the core–EP inverse and the weak group inverse, are particular kinds of the minimal rank weak Drazin inverse [21]. Dually, a right weak Drazin inverse of A is a solution to $A^{k+1}X = A^k$; and a minimal rank right weak Drazin inverse of A [3] is a solution to

$$A^{k+1}X = A^k \quad \text{and} \quad rank(X) = rank(A^D).$$

The weak DMP and weak MPD inverses were recently introduced in [14] as extensions of DMP and MPD inverses [10]. If X is an arbitrary but fixed minimal rank weak Drazin inverse of $A \in \mathbb{C}^{n\times n}$, the weak MPD inverse of A is given by

$$A^{w,\dagger,D} = A^\dagger XA.$$

For an arbitrary but fixed minimal rank right weak Drazin inverse Z of A, the weak DMP inverse of A is represented as

$$A^{w,D,\dagger} = AZA^\dagger.$$

Motivated by significant properties of the minimal rank weak Drazin inverse, MPCEP and *CEPMP inverses, we continue to investigate these topics. In particular, we investigate generalized versions of the systems (1) and (2) in terms of a minimal rank weak Drazin inverse or a minimal rank right weak Drazin inverse. Solving new systems of matrix equations, in a natural way, we define new kinds of generalized inverses, which involves the DMP, MPD, MPCEP and *CEPMP inverses as special types. Thus, we introduce wider classes of generalized inverses. The next research directions are given in this paper.

(1) Since the core–EP inverse is a particular kind of minimal rank weak Drazin inverse, we generalize the system (1) using a minimal rank weak Drazin inverse of A instead of the core–EP inverse $A^{\oplus}$. As a solution of new weakened

system, we define new generalized inverse which is called the weak MPCEP inverse. We observe that MPCEP and MPD inverses are special cases of the weak MPCEP inverse.

(2) Replacing the *core–EP inverse $A_{\textcircled{\dagger}}$ with a minimal rank right weak Drazin inverse of A, we generalize the system (2) and introduce the weak *CEPMP inverse. The *CEPMP and DMP inverses are particular kinds of the weak *CEPMP inverse.

(3) Many characterizations of weak MPCEP and *CEPMP inverses are developed.

(4) Some formulae for weak MPCEP inverse are given.

(5) Perturbation results of weak MPCEP and *CEPMP inverses are proposed.

(6) We solve certain linear equations applying weak MPCEP and *CEPMP inverses.

(7) Consequently, we obtain new and known properties of the MPCEP, *CEPMP, MPD and DMP inverses as well as of two for the first time mentioned generalized inverses.

(8) In order to illustrate our results, numerical examples are presented.

This is the content of our paper. Section 2 involves definitions and characterizations of weak MPCEP and *CEPMP inverses. Formulae of weak MPCEP inverses are given in Section 3. Perturbation formulae and perturbation bounds of weak MPCEP and *CEPMP inverses are established in Section 4. Solvability of some linear equations is proved in Section 5 in terms of weak MPCEP and *CEPMP inverses. As a consequence, the major application of the Moore-Penrose inverse in solving linear equation is obtained.

2 Weak MPCEP and *CEPMP inverses

Based on a minimal rank weak Drazin inverse or minimal rank right weak Drazin inverse instead of the core–EP inverse and *core–EP inverse, we consider generalized versions of the systems (1) and (2). As solutions of new weaker systems of matrix equations, we naturally define two new types of generalized inverses.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A. Then*

(a) *$Y = A^\dagger AX$ is uniquely determined solution to the system of matrix equations*

$$YAY = Y, \quad AY = AX \quad \text{and} \quad YA = A^\dagger AXA. \tag{3}$$

(b) *$Y = ZAA^\dagger$ is uniquely determined solution to the system of matrix equations*

$$YAY = Y, \quad YA = ZA \quad \textit{and} \quad AY = AZAA^\dagger.$$

Proof. (a) According to [21, Theorem 2.1], a minimal rank weak Drazin inverse X of A satisfies $XAX = X$. If $Y = A^\dagger AX$, we get $YA = A^\dagger AXA$, $AY = (AA^\dagger A)X = AX$ and

$$YAY = YAX = A^\dagger A(XAX) = A^\dagger AX = Y,$$

that is, (3) has a solution $Y = A^\dagger AX$.

A solution Y of (3) is uniquely determined because

$$Y = (YA)Y = A^\dagger AX(AY) = A^\dagger A(XAX) = A^\dagger AX.$$

(b) This part follows similarly by properties of a minimal rank right weak Drazin inverse Z of A given in [14, Lemma 1.1]. □

Definition 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A. Then*

(a) *the weak MPCEP inverse of A is introduced as*

$$A^{w,\dagger,\oplus} = A^\dagger AX.$$

(b) *the weak *CEPMP inverse of A is introduced as*

$$A_{w,\oplus,\dagger} = ZAA^\dagger.$$

Classes of weak MPCEP and *CEPMP inverses contain as particular types the next well-known generalized inverses:

- for $X = A^D$, the weak MPCEP inverse reduces to the MPD inverse $A^\dagger AA^D = A^{\dagger,D}$ [10];
- if $Z = A^D$, the weak *CEPMP inverse is equal to the DMP inverse $A^D AA^\dagger = A^{D,\dagger}$ [10];
- in the case that $X = A^{D,\dagger}$ (or $Z = A^{\dagger,D}$), the weak MPCEP (or *CEPMP) inverse coincides with the CMP inverse $A^\dagger AA^{D,\dagger} = A^\dagger AA^D AA^\dagger = A^{c,\dagger}$ [11];
- when $X = A^\oplus$, the weak MPCEP inverse becomes the MPCEP inverse $A^\dagger AA^\oplus = A^{\dagger,\oplus}$ [4];

- taking $Z = A_{\textcircled{\dagger}}$, the weak *CEPMP inverse is the *CEPMP inverse $A_{\textcircled{\dagger}}AA^\dagger = A_{\textcircled{\dagger},\dagger}$ [4];
- for $X = A^{\textcircled{w}}$, the weak MPCEP inverse is equal to $A^\dagger AA^{\textcircled{w}}$, which represents, by Theorem 2.1, the unique solution to the system

$$YAY = Y, \quad AY = AA^{\textcircled{w}} \quad \text{and} \quad YA = A^\dagger AA^{\textcircled{w}}A;$$

- if $X = A^{\textcircled{w}_m}$, where $m \in \mathbb{N}$, the weak MPCEP inverse of A reduces to $A^\dagger AA^{\textcircled{w}_m}$, i.e. uniquely determined solution to

$$YAY = Y, \quad AY = AA^{\textcircled{w}_m} \quad \text{and} \quad YA = A^\dagger AA^{\textcircled{w}_m}A.$$

Remark that $A^\dagger AA^{\textcircled{w}}$ and $A^\dagger AA^{\textcircled{w}_m}$ represent new generalized inverses of A considering here for the first time in literature.

In the following example, we observe that the weak MPCEP inverse is different form the well-known generalized inverses and some of them are its special cases.

Example 2.1. *Considering*

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

it follows $\text{ind}(A) = 2$*,*

$$A^D = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{32} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$A^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad A^{\textcircled{w}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The minimal rank weak Drazin inverse X of A has the form

$$X = \begin{bmatrix} \frac{1}{4} & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

where $x_1, x_2 \in \mathbb{C}$. Thus, the weak MPCEP inverse of A is given by

$$Y = A^\dagger AX = \begin{bmatrix} \frac{1}{5} & \frac{4}{5}x_1 & \frac{4}{5}x_2 \\ \frac{1}{10} & \frac{2}{5}x_1 & \frac{2}{5}x_2 \\ 0 & 0 & 0 \end{bmatrix},$$

and the weak MPD inverse is

$$A^{w,\dagger,D} = A^\dagger XA = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{5}x_1 \\ \frac{1}{10} & \frac{1}{20} & \frac{1}{10}x_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the case that $x_1 = \frac{1}{8}$ *and* $x_2 = \frac{1}{32}$*, the minimal rank weak Drazin inverse* X *reduces to the Drazin inverse* A^D *and the weak MPCEP inverse* Y *coincides with the MPD inverse*

$$A^{\dagger,D} = A^\dagger AA^D = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{40} \\ \frac{1}{10} & \frac{1}{20} & \frac{1}{80} \\ 0 & 0 & 0 \end{bmatrix}.$$

When $x_1 = x_2 = 0$*, the minimal rank weak Drazin inverse* X *becomes the core–EP inverse* $A^{\textcircled{\dagger}}$ *and the weak MPCEP inverse* Y *is equal to the MPCEP inverse*

$$A^{\dagger,\textcircled{\dagger}} = A^\dagger AA^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $x_1 = \frac{1}{8}$ *and* $x_2 = 0$*,* X *is equal to the weak group inverse* $A^{\textcircled{w}}$ *and the weak MPCEP inverse* Y *is*

$$A^\dagger AA^{\textcircled{w}} = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{1}{20} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The weak MPCEP inverse can be also considered as solution of the following systems of matrix equations.

Theorem 2.2. *Let* $A \in \mathbb{C}^{n\times n}$*,* $k = \mathrm{ind}(A)$ *and* X *be a minimal rank weak Drazin inverse of* A*. The next statements are equivalent for* $Y \in \mathbb{C}^{n\times n}$*:*

(i) $Y = A^\dagger AX$;

(ii) $YAY = Y$, $AYA = AXA$, $AY = AX$ *and* $YA = A^\dagger AXA$;

(iii) $YAY = Y$, $AY = AX$ *and* $YA^k = A^\dagger A^k$;

(iv) $Y = A^\dagger AXAY$ *and* $AY = AX$;

(v) $Y = A^\dagger AXAY$ *and* $A^\dagger AY = A^\dagger AX$;

(vi) $Y = A^\dagger AXAY$ *and* $A^*AY = A^*AX$;

(vii) $Y = A^\dagger AXAY$ *and* $XAY = X$;

(viii) $Y = YAX$ *and* $YA = A^\dagger AXA$;

(ix) $Y = YAX$ *and* $YAA^\dagger = A^\dagger AXAA^\dagger$;

(x) $Y = YAX$ *and* $YAA^* = A^\dagger AXAA^*$;

(xi) $Y = YAX$ *and* $YAX = A^\dagger AX$;

(xii) $Y = YAX$ *and* $YA^k = A^\dagger A^k$;

(xiii) $Y = YAX$ *and* $YAA^D = A^{\dagger,D}$;

(xiv) $Y = A^\dagger AY$ *and* $AY = AX$;

(xv) $Y = A^\dagger AY$ *and* $A^\dagger AY = A^\dagger AX$;

(xvi) $Y = A^\dagger AY$ *and* $A^*AY = A^*AX$;

(xvii) $YAXAY = Y$, $AXAYAXA = AXA$, $AXAY = AX$ *and* $YAXA = A^\dagger AXA$;

(xviii) $YAXAY = Y$, $AXAY = AX$ *and* $YAXA = A^\dagger AXA$.

Proof. (i) ⇔ (ii): Theorem 2.1 implies this equivalence.

(i) ⇒ (iii): Recall that, by [21, Theorem 2.1], $XA^{k+1} = A^k$. For $Y = A^\dagger AX$, we get

$$YA^k = A^\dagger A(XA^{k+1})A^D = A^\dagger(A^{k+1}A^D) = A^\dagger A^k.$$

(iii) ⇒ (i): Applying [21, Theorem 2.1], we have $X = AX^2 = A^2X^3 = \cdots = A^kX^{k+1}$. The assumptions $YAY = Y$, $AY = AX$ and $YA^k = A^\dagger A^k$ yield

$$Y = Y(AY) = YAX = (YA^k)AX^{k+1} = A^\dagger A(A^kX^{k+1}) = A^\dagger AX.$$

(ii) ⇒ (iv): One can observe that $YAY = Y$ and $YA = A^\dagger AXA$ give $Y = (YA)Y = A^\dagger AXAY$.

(iv) ⇒ (v): It is evident.

(v) ⇒ (vi): From $A^\dagger AY = A^\dagger AX$, we obtain $A^*AY = A^*A(A^\dagger AY) = (A^*AA^\dagger)AX = A^*AX$.

(vi) ⇒ (vii): The hypothesis $A^*AY = A^*AX$ implies

$$XAY = XAA^\dagger AY = X(A^\dagger)^*(A^*AY) = X(A^\dagger)^*A^*AX = XAX = X.$$

(vii) ⇒ (i): Using $Y = A^\dagger AXAY$ and $XAY = X$, we have that $Y = A^\dagger A(XAY) = A^\dagger AX$.

This proof can be finished in an analogy manner. □

In the particular case that $X = A^D$, Theorem 2.2 implies characterizations of the MPD inverse and recovers [17, Corollary 3.1]. When $X = A^{\textcircled{\dagger}}$ in Theorem 2.2, we characterize the MPCEP inverse and get [17, Corollary 3.3].

If $X = A^{\textcircled{w}_m}$ in Theorem 2.2, a list of equivalent conditions for $Y = A^\dagger A A^{\textcircled{w}_m}$ can be obtained.

Corollary 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ and $m \in \mathbb{N}$. The following statements are equivalent for $Y \in \mathbb{C}^{n\times n}$:*

(i) $Y = A^\dagger A A^{\textcircled{w}_m}$;

(ii) $YAY = Y$, $AYA = AA^{\textcircled{w}_m}A$, $AY = AA^{\textcircled{w}_m}$ *and* $YA = A^\dagger A A^{\textcircled{w}_m} A$;

(iii) $YAY = Y$, $AY = AA^{\textcircled{w}_m}$ *and* $YA^k = A^\dagger A^k$;

(iv) $Y = A^\dagger A A^{\textcircled{w}_m} AY$ *and* $AY = AA^{\textcircled{w}_m}$;

(v) $Y = A^\dagger A A^{\textcircled{w}_m} AY$ *and* $A^\dagger AY = A^\dagger A A^{\textcircled{w}_m}$;

(vi) $Y = A^\dagger A A^{\textcircled{w}_m} AY$ *and* $A^* AY = A^* A A^{\textcircled{w}_m}$;

(vii) $Y = A^\dagger A A^{\textcircled{w}_m} AY$ *and* $A^{\textcircled{w}_m} AY = A^{\textcircled{w}_m}$;

(viii) $Y = YAA^{\textcircled{w}_m}$ *and* $YA = A^\dagger A A^{\textcircled{w}_m} A$;

(ix) $Y = YAA^{\textcircled{w}_m}$ *and* $YAA^\dagger = A^\dagger A A^{\textcircled{w}_m} AA^\dagger$;

(x) $Y = YAA^{\textcircled{w}_m}$ *and* $YAA^* = A^\dagger A A^{\textcircled{w}_m} AA^*$;

(xi) $Y = YAA^{\textcircled{w}_m}$ *and* $YAA^{\textcircled{w}_m} = A^\dagger A A^{\textcircled{w}_m}$;

(xii) $Y = YAA^{\textcircled{w}_m}$ *and* $YA^k = A^\dagger A^k$;

(xiii) $Y = YAA^{\textcircled{w}_m}$ *and* $YAA^D = A^{\dagger,D}$;

(xiv) $Y = A^\dagger AY$ *and* $AY = AA^{\textcircled{w}_m}$;

(xv) $Y = A^\dagger AY$ *and* $A^\dagger AY = A^\dagger A A^{\textcircled{w}_m}$;

(xvi) $Y = A^\dagger AY$ *and* $A^* AY = A^* A A^{\textcircled{w}_m}$;

(xvii) $YAA^{\textcircled{w}_m} AY = Y$, $AA^{\textcircled{w}_m} AYAA^{\textcircled{w}_m} A = AA^{\textcircled{w}_m} A$, $AA^{\textcircled{w}_m} AY = AA^{\textcircled{w}_m}$ *and* $YAA^{\textcircled{w}_m} A = A^\dagger A A^{\textcircled{w}_m} A$;

(xviii) $YAA^{\textcircled{w}_m} AY = Y$, $AA^{\textcircled{w}_m} AY = AA^{\textcircled{w}_m}$ *and* $YAA^{\textcircled{w}_m} A = A^\dagger A A^{\textcircled{w}_m} A$.

Notice that $AA^{ⓦ_m}$ in Corollary 2.1 can be replaced with $(A^{\textcircled{\dagger}})^m A^m$ to obtain more characterizations of $Y = A^\dagger AA^{ⓦ_m}$. For $m = 1$ in Corollary 2.1, characterizations of $Y = A^\dagger AA^{ⓦ_m}$ can be given.

Necessary and sufficient conditions for a given matrix to be the weak *CEPMP inverse can be established as Theorem 2.2.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$ and Z be a minimal rank right weak Drazin inverse of A. The following statements are equivalent for $Y \in \mathbb{C}^{n\times n}$:*

(i) $Y = ZAA^\dagger$;

(ii) $YAY = Y$, $AYA = AZA$, $YA = ZA$ and $AY = AZAA^\dagger$;

(iii) $YAY = Y$, $YA = ZA$ and $A^kY = A^kA^\dagger$;

(iv) $Y = YAZAA^\dagger$ and $YA = ZA$;

(v) $Y = YAZAA^\dagger$ and $YAA^\dagger = ZAA^\dagger$;

(vi) $Y = YAZAA^\dagger$ and $YAA^* = ZAA^*$;

(vii) $Y = YAZAA^\dagger$ and $YAZ = Z$;

(viii) $Y = ZAY$ and $AY = AZAA^\dagger$;

(ix) $Y = ZAY$ and $A^\dagger AY = A^\dagger AZAA^\dagger$;

(x) $Y = ZAY$ and $A^*AY = A^*AZAA^\dagger$;

(xi) $Y = ZAY$ and $ZAY = ZAA^\dagger$;

(xii) $Y = ZAY$ and $A^kY = A^kA^\dagger$;

(xiii) $Y = ZAY$ and $A^DAY = A^{D,\dagger}$;

(xiv) $Y = YAA^\dagger$ and $YA = ZA$;

(xv) $Y = YAA^\dagger$ and $YAA^\dagger = ZAA^\dagger$;

(xvi) $Y = YAA^\dagger$ and $YAA^* = ZAA^*$;

(xvii) $YAZAY = Y$, $AZAYAZA = AZA$, $YAZA = ZA$ and $AZAY = AZAA^\dagger$;

(xviii) $YAZAY = Y$, $YAZA = ZA$ and $AZAY = AZAA^\dagger$.

Especially, for $Z = A^D$ and $Z = A_{\textcircled{\dagger}}$ in Theorem 2.3, we can verify characterizations of the DMP inverse and the *CEPMP inverse proposed in [17, Corollary 2.1 and Corollary 2.3].

Theorem 2.1 gives that weak MPCEP and *CEPMP inverses are outer inverses of A, and, by Theorem 2.2 and Theorem 2.3, they are both outer and inner inverses of AXA and AZA, respectively. We consider ranges and null spaces of weak MPCEP and *CEPMP inverses and of projections defining by them. For subspaces G and H, $P_{G,H}$ marks the projector onto G along H; and P_G is the orthogonal projector onto G.

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A.*

(a) *For $Y = A^\dagger AX$, we have*

(i) $AY = P_{\mathcal{R}(A^k),\mathcal{N}(X)}$;

(ii) $YA = P_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(XA)}$;

(iii) $Y = A^{(2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(X)} = (AXA)^{(1,2)}_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(X)}$.

(b) *For $Y = ZAA^\dagger$, we have*

(i) $AY = P_{\mathcal{R}(AZ),\mathcal{N}(A^kA^\dagger)}$;

(ii) $YA = P_{\mathcal{R}(Z),\mathcal{N}(A^k)}$;

(iii) $Y = A^{(2)}_{\mathcal{R}(Z),\mathcal{N}(A^kA^\dagger)} = (AZA)^{(1,2)}_{\mathcal{R}(Z),\mathcal{N}(A^kA^\dagger)}$.

Proof. (a) (i) Theorem 2.1 gives $AY = AX$. Because $\mathcal{R}(X) = \mathcal{R}(A^k)$, it follows $\mathcal{R}(AY) = \mathcal{R}(AX) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k)$ and, by $XAX = X$, $\mathcal{N}(AY) = \mathcal{N}(AX) = \mathcal{N}(X)$.

(ii) Since $YA = A^\dagger AXA$, $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we obtain

$$\mathcal{R}(YA) = \mathcal{R}(A^\dagger AXA) = \mathcal{R}(A^\dagger AX) = \mathcal{R}(A^\dagger AA^k) = \mathcal{R}(A^\dagger A^k).$$

We conclude that $\mathcal{N}(YA) = \mathcal{N}(A^\dagger AXA) = \mathcal{N}(AXA) = \mathcal{N}(XA)$.

(iii) By parts (ii) and (iii), note that $\mathcal{N}(Y) = \mathcal{N}(AY) = \mathcal{N}(X)$ and $\mathcal{R}(Y) = \mathcal{R}(YA) = \mathcal{R}(A^\dagger A^k)$. Theorem 2.2 implies that Y is both inner and outer inverse of AXA.

Similarly, we verify part (b). □

Using Urquhart formula [2, Theorem 13] and Lemma 2.1, we can get new formulae of weak MPCEP and *CEPMP inverses.

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A. Then*

$$A^{\dagger}AX = A^{\dagger}A^{k}(XA^{k})^{\dagger}X$$

and

$$ZAA^{\dagger} = Z(A^{k}Z)^{\dagger}A^{k}A^{\dagger}.$$

We develop new and recover well-known expressions for particular types of weak MPCEP and *CEPMP inverses applying Lemma 2.1 and Corollary 2.2.

Corollary 2.3. *If $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$, we have*

(i) $A^{\dagger,D} = (AA^{D}A)^{(1,2)}_{\mathcal{R}(A^{\dagger}A^{k}),\mathcal{N}(A^{k})} = A^{\dagger}A^{k}(A^{D}A^{k})^{\dagger}A^{D}$;

(ii) $A^{\dagger,\oplus} = (AA^{\oplus}A)^{(1,2)}_{\mathcal{R}(A^{\dagger}A^{k}),\mathcal{N}((A^{k})^{*})} = A^{\dagger}A^{k}(A^{\oplus}A^{k})^{\dagger}A^{\oplus} = A^{\dagger}A^{k}(A^{D}A^{k})^{\dagger}A^{\oplus}$;

(iii) $A^{\dagger}AA^{\oplus_m} = A^{(2)}_{\mathcal{R}(A^{\dagger}A^{k}),\mathcal{N}((A^{k})^{*}A^{m})} = (AA^{\oplus_m}A)^{(1,2)}_{\mathcal{R}(A^{\dagger}A^{k}),\mathcal{N}((A^{k})^{*}A^{m})} = A^{\dagger}A^{k}(A^{\oplus_m}A^{k})^{\dagger}A^{\oplus_m} = A^{\dagger}A^{k}(A^{D}A^{k+m})^{\dagger}A^{\oplus_m}$, *where $m \in \mathbb{N}$;*

(iv) $A^{D,\dagger} = (AA^{D}A)^{(1,2)}_{\mathcal{R}(A^{k}),\mathcal{N}(A^{k}A^{\dagger})} = A^{D}(A^{k}A^{D})^{\dagger}A^{k}A^{\dagger}$;

(v) $A_{\oplus,\dagger} = (AA_{\oplus}A)^{(1,2)}_{\mathcal{R}((A^{k})^{*}),\mathcal{N}(A^{k}A^{\dagger})} = A_{\oplus}(A^{k}A_{\oplus})^{\dagger}A^{k}A^{\dagger} = A_{\oplus}(A^{k}A^{D})^{\dagger}A^{k}A^{\dagger}$.

Several equivalent conditions for MPCEP and *CEPMP inverses of A to be in the set $A\{1\}$, are investigated. Also, one can see that the weak MPCEP inverse $A^{\dagger}AX$ and weak MPD inverse $A^{\dagger}XA$ coincide if and only if X is equal to A^{D}.

Theorem 2.4. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A. Then*

(i) $A^{\dagger}AX \in A\{1\}$ *if and only if* $X \in A\{1\}$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(XA)$ *if and only if* $\mathcal{R}(A) = \mathcal{R}(A^{k})$ *if and only if* $k \le 1$;

(ii) $ZAA^{\dagger} \in A\{1\}$ *if and only if* $Z \in A\{1\}$ *if and only if* $\mathcal{R}(A) = \mathcal{R}(AZ)$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(A^{k})$ *if and only if* $k \le 1$;

(iii) $A^{\dagger}AX = A^{\dagger}XA$ *if and only if* $AX = XA$ *if and only if* $X = A^{D}$.

Proof. (i) The first equivalence is clear by $AA^\dagger AXA = AXA$. Further, by $\mathcal{R}(X) = \mathcal{R}(A^k)$,

$$\begin{aligned} A = AXA &\Leftrightarrow A(I - XA) = 0 \Leftrightarrow \mathcal{R}(I - XA) \subseteq \mathcal{N}(A) \\ &\Leftrightarrow \mathcal{N}(XA) \subseteq \mathcal{N}(A) \Leftrightarrow \mathcal{N}(XA) = \mathcal{N}(A) \\ &\Leftrightarrow (I - AX)A = 0 \Leftrightarrow \mathcal{R}(A) \subseteq \mathcal{N}(I - AX) \\ &\Leftrightarrow \mathcal{R}(A) \subseteq \mathcal{R}(AX) \Leftrightarrow \mathcal{R}(A) = \mathcal{R}(AX) \\ &\Leftrightarrow \mathcal{R}(A) = \mathcal{R}(A^k). \end{aligned}$$

Part (ii) can be checked similarly.

(iii) It is clear, by $A^\dagger AX = A^\dagger XA$ and $X = AX^2$, that

$$AX = A(A^\dagger AX) = AA^\dagger XA = (AA^\dagger A)X^2A = (AX^2)A = XA.$$

The converse is obvious. □

Weak MPD and DMP inverses can be presented as solutions of the next equations with constrain.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$, X be a minimal rank weak Drazin inverse of A and Z be a minimal rank right weak Drazin inverse of A. Then*

(i) $Y = A^\dagger AX$ *is uniquely determined solution to*

$$AY = P_{\mathcal{R}(A^k),\mathcal{N}(X)} \quad \textit{and} \quad \mathcal{R}(Y) \subseteq \mathcal{R}(A^*); \tag{4}$$

(ii) $Y = A^\dagger AX$ *is uniquely determined solution to*

$$YA = P_{\mathcal{R}(A^\dagger A^k),\mathcal{N}(XA)} \quad \textit{and} \quad \mathcal{R}(Y^*) \subseteq \mathcal{R}(X^*); \tag{5}$$

(iii) $Y = ZAA^\dagger$ *is uniquely determined solution to*

$$AY = P_{\mathcal{R}(AZ),\mathcal{N}(A^kA^\dagger)} \quad \textit{and} \quad \mathcal{R}(Y) \subseteq \mathcal{R}(AZ);$$

(iv) $Y = ZAA^\dagger$ *is uniquely determined solution to*

$$YA = P_{\mathcal{R}(Z),\mathcal{N}(A^k)} \quad \textit{and} \quad \mathcal{R}(Y^*) \subseteq \mathcal{R}(A).$$

Proof. (i) We deduce, using Lemma 2.1, that $Y = A^\dagger AX$ is a solution to (4).

If (4) has two solutions Y and U, then $A(Y-U) = 0$ and $\mathcal{R}(Y-U) \subseteq \mathcal{R}(A^*)$. So, $\mathcal{R}(Y-U) \subseteq \mathcal{N}(A) \cap \mathcal{R}(A^*) = \{0\}$, that is, $Y = A^\dagger AX$ is the unique solution to (4).

(ii) Notice that $Y^* = X^*A^\dagger A$ and Lemma 2.1 imply that (5) has a solution $Y = A^\dagger AX$.

For Y and U as two solutions of (5), it follows $A^*(Y^* - U^*) = 0$ and $\mathcal{R}(Y^* - U^*) \subseteq \mathcal{R}(X^*) = \mathcal{R}(X^*A^*)$. Thus, $\mathcal{R}(Y^* - U^*) \subseteq \mathcal{N}(A^*) \cap \mathcal{R}(X^*A^*) \subseteq \mathcal{N}(X^*A^*) \cap \mathcal{R}(X^*A^*) = \{0\}$, i.e. (5) has the unique solution $Y = A^\dagger AX$.

The rest can be proved analogously. □

New characterizations of $Y = A^\dagger AX$ are presented in the next result.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. The next statements are equivalent for $Y \in \mathbb{C}^{n\times n}$:*

(i) $Y = A^\dagger AX$;

(ii) $\mathcal{R}(Y) = \mathcal{R}(A^\dagger A^k)$ *and* $AY = AX$;

(iii) $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k)$ *and* $AY = AX$;

(iv) $\mathcal{R}(Y^*) = \mathcal{R}(X^*)$ *and* $YA = A^\dagger AXA$;

(v) $\mathcal{R}(Y^*) \subseteq \mathcal{R}(X^*)$ *and* $YA = A^\dagger AXA$.

Proof. (i) ⇒ (ii): Lemma 2.1 and Theorem 2.2 yield this part.

(ii) ⇒ (iii): Clearly.

(iii) ⇒ (i): The hypothesis $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k)$ gives $Y = A^\dagger A^k H$, for some $H \in \mathbb{C}^{n\times n}$. Using $AY = AX$, we observe that $Y = A^\dagger A(A^\dagger A^k H) = A^\dagger(AY) = A^\dagger AX$.

In a similar manner, the proof can be completed. □

Theorem 2.6 can imply characterizations of special kinds of weak MPCEP inverses and we give consequences related to the MPCEP inverse and $Y = A^\dagger A A^{\textcircled{w}_m}$.

Corollary 2.4. *Let $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{ind}(A)$. The next statements are equivalent for $Y \in \mathbb{C}^{n\times n}$:*

(i) $Y = A^\dagger AA^{\textcircled{\dagger}} (= A^{\dagger,\textcircled{\dagger}})$;

(ii) $\mathcal{R}(Y) = \mathcal{R}(A^\dagger A^k)$ *and* $AY = AA^{\textcircled{\dagger}}$;

(iii) $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k)$ *and* $AY = AA^{\textcircled{\dagger}}$;

(iv) $\mathcal{R}(Y^*) = \mathcal{R}(A^k)$ *and* $YA = A^\dagger AA^{\textcircled{\dagger}}A$;

(v) $\mathcal{R}(Y^*) \subseteq \mathcal{R}(A^k)$ *and* $YA = A^\dagger AA^{\textcircled{\dagger}}A$.

Corollary 2.5. *Let $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ and $m \in \mathbb{N}$. The next statements are equivalent for $Y \in \mathbb{C}^{n\times n}$:*

(i) $Y = A^\dagger A A^{\circledw_m}$;

(ii) $\mathcal{R}(Y) = \mathcal{R}(A^\dagger A^k)$ *and* $AY = AA^{\circledw_m}$;

(iii) $\mathcal{R}(Y) \subseteq \mathcal{R}(A^\dagger A^k)$ *and* $AY = AA^{\circledw_m}$;

(iv) $\mathcal{R}(Y^*) = \mathcal{R}((A^m)^* A^k)$ *and* $YA = A^\dagger A A^{\circledw_m} A$;

(v) $\mathcal{R}(Y^*) \subseteq \mathcal{R}((A^m)^* A^k)$ *and* $YA = A^\dagger A A^{\circledw_m} A$.

We establish the following characterizations of $Y = ZAA^\dagger$ as Theorem 2.6.

Theorem 2.7. *Let* $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ *and* Z *be a minimal rank right weak Drazin inverse of* A. *The next statements are equivalent for* $Y \in \mathbb{C}^{n\times n}$:

(i) $Y = ZAA^\dagger$;

(ii) $\mathcal{R}(Y) = \mathcal{R}(Z)$ *and* $AY = AZAA^\dagger$;

(iii) $\mathcal{R}(Y) \subseteq \mathcal{R}(Z)$ *and* $AY = AZAA^\dagger$;

(iv) $\mathcal{R}(Y^*) = \mathcal{R}(A)$ *and* $YA = ZA$;

(v) $\mathcal{R}(Y^*) \subseteq \mathcal{R}(A)$ *and* $YA = ZA$.

Theorem 2.7 can give characterizations of the DMP and *CEPMP inverses.

3 Formulae for weak MPCEP inverse

This section contains significant formulae of weak MPCEP inverse.

Firstly, we express the weak MPCEP and *CEPMP inverses using MPD and DMP inverses.

Lemma 3.1. *Let* $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$, X *be a minimal rank weak Drazin inverse of* A *and* Z *be a minimal rank right weak Drazin inverse of* A. *Then*

$$A^\dagger AX = A^{\dagger,D} AX$$

and

$$ZAA^\dagger = ZAA^{D,\dagger}.$$

Proof. For some $U \in \mathbb{C}^{n\times n}$, recall that $\mathcal{R}(X) = \mathcal{R}(A^D)$ gives $X = A^D AU = A^D AX$. Now, $A^\dagger AX = (A^\dagger AA^D)AX = A^{\dagger,D} AX$. The second formula follows in a same manner. □

The canonical forms of the minimal rank weak Drazin inverse and Moore-Penrose inverse were given in [21] and [5], respectively, for a square matrix decomposed as in [19].

Lemma 3.2. [19] *If* $A \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = k$ *and* $rank(A^k) = t$, *then*

$$A = U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} U^*, \tag{6}$$

where $U \in \mathbb{C}^{n\times n}$ *is unitary,* $A_1 \in \mathbb{C}^{t\times t}$ *is invertible upper-triangular and* $A_3 \in \mathbb{C}^{(n-t)\times(n-t)}$ *is a nilpotent of index* k. *In addition, a minimal rank weak Drazin inverse* X *of* A *is represented by* [21]:

$$X = U \begin{bmatrix} A_1^{-1} & V \\ 0 & 0 \end{bmatrix} U^*, \tag{7}$$

where $V \in \mathbb{C}^{t\times(n-t)}$. *Furthermore, the Moore-Penrose inverse of* A *is given as* [5]:

$$A^\dagger = U \begin{bmatrix} A_1^*\triangle & -A_1^*\triangle A_2 A_3^\dagger \\ (I - A_3^\dagger A_3)A_2^*\triangle & A_3^\dagger - (I - A_3^\dagger A_3)A_2^*\triangle A_2 A_3^\dagger \end{bmatrix} U^*, \tag{8}$$

where $\triangle = (A_1A_1^* + A_2(I - A_3^\dagger A_3)A_2^*)^{-1}$.

We now propose the canonical form for the weak MPCEP inverse.

Lemma 3.3. *If* $A \in \mathbb{C}^{n\times n}$ *with* $k = \mathrm{ind}(A)$ *is given by* (6) *and* X *is a minimal rank weak Drazin inverse of* A *represented by* (7), *then*

$$Y = A^\dagger AX = U \begin{bmatrix} A_1^*\triangle & A_1^*\triangle A_1 V \\ (I - A_3^\dagger A_3)A_2^*\triangle & (I - A_3^\dagger A_3)A_2^*\triangle A_1 V \end{bmatrix} U^*, \tag{9}$$

where $\triangle = (A_1A_1^* + A_2(I - A_3^\dagger A_3)A_2^*)^{-1}$.

Proof. The equalities (6), (7) and (8) yield

$$\begin{aligned} Y &= A^\dagger AX \\ &= U \begin{bmatrix} A_1^*\triangle A_1 & A_1^*\triangle A_2(I - A_3^\dagger A_3) \\ (I - A_3^\dagger A_3)A_2^*\triangle A_1 & A_3^\dagger A_3 + (I - A_3^\dagger A_3)A_2^*\triangle A_2(I - A_3^\dagger A_3) \end{bmatrix} U^* \\ &\times X \\ &= U \begin{bmatrix} A_1^*\triangle & A_1^*\triangle A_1 V \\ (I - A_3^\dagger A_3)A_2^*\triangle & (I - A_3^\dagger A_3)A_2^*\triangle A_1 V \end{bmatrix} U^*. \end{aligned}$$

□

Based on projections $F = I - AY$ and $G = I - YA$, the next expressions of the weak MPCEP inverse Y can be established.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. For $Y = A^\dagger AX$, $F = I - AY$ and $G = I - YA$, $A \pm F$ are nonsingular and*

$$Y = (I - G)(A \pm F)^{-1}(I - F). \tag{10}$$

Proof. Assume that A and Y, respectively, are represented as in (6) and (9). Then

$$\begin{aligned} F &= I - AY = I - U \begin{bmatrix} I & A_1V \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} 0 & -A_1V \\ 0 & I \end{bmatrix} U^* \end{aligned}$$

and

$$\begin{aligned} I - G &= YA \\ &= U \begin{bmatrix} A_1^*\triangle A_1 & A_1^*\triangle(A_2 + A_1VA_3) \\ (I - A_3^\dagger A_3)A_2^*\triangle A_1 & (I - A_3^\dagger A_3)A_2^*\triangle(A_2 + A_1VA_3) \end{bmatrix} U^*. \end{aligned}$$

Since A_1 and $A_3 \pm I$ are nonsingular, we deduce that

$$A \pm F = U \begin{bmatrix} A_1 & A_2 \mp A_1V \\ 0 & A_3 \pm I \end{bmatrix} U^*$$

is nonsingular too and

$$(A \pm F)^{-1} = U \begin{bmatrix} A_1^{-1} & -A_1^{-1}(A_2 \mp A_1V)(A_3 \pm I)^{-1} \\ 0 & (A_3 \pm I)^{-1} \end{bmatrix} U^*.$$

Now, (10) is satisfied by

$$\begin{aligned} & (I - G)(A \pm F)^{-1}(I - F) = \\ = \;& (I - G)U \begin{bmatrix} A_1^{-1} & V \\ 0 & 0 \end{bmatrix} U^* \\ = \;& U \begin{bmatrix} A_1^*\triangle & A_1^*\triangle A_1V \\ (I - A_3^\dagger A_3)A_2^*\triangle & (I - A_3^\dagger A_3)A_2^*\triangle A_1V \end{bmatrix} U^* \\ = \;& Y. \end{aligned}$$

□

The following example is given to illustrate Theorem 3.1.

Example 3.1. *For the matrix A as in Example 2.1, we have*

$$I-F=AY=\begin{bmatrix}1 & 4x_1 & 4x_2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad F=I-AY=\begin{bmatrix}0 & -4x_1 & -4x_2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

and

$$I-G=YA=\begin{bmatrix}\frac{4}{5} & \frac{2}{5} & \frac{4}{5}x_1\\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5}x_1\\ 0 & 0 & 0\end{bmatrix}.$$

Further

$$A+F=\begin{bmatrix}4 & 2-4x_1 & -4x_2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\quad A-F=\begin{bmatrix}4 & 2+4x_1 & 4x_2\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix},$$

$$(A+F)^{-1}=\begin{bmatrix}\frac{1}{4} & -\frac{1}{2}+x_1 & \frac{1}{2}-x_1+x_2\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

and

$$(A-F)^{-1}=\begin{bmatrix}\frac{1}{4} & \frac{1}{2}+x_1 & \frac{1}{2}+x_1+x_2\\ 0 & -1 & -1\\ 0 & 0 & -1\end{bmatrix}.$$

Using elementary calculations, we get $(I-G)(A\pm F)^{-1}(I-F)=Y$.

The set of all weak MPCEP inverses of A can be described in the next manners.

Theorem 3.2. *Let $A\in\mathbb{C}^{n\times n}$, $k=\operatorname{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. The set of all weak MPCEP inverses of A is expressed as:*

(i) $\{A^{\dagger,D}+A^{\dagger,D}AE(I-A^DA):\ E\in\mathbb{C}^{n\times n}\}$;

(ii) $\{A^{\dagger}AX+A^{\dagger}AXF(I-XA):\ F\in\mathbb{C}^{n\times n}\}$;

(iii) $\{A^{\dagger}AX+A^{\dagger}AXF(I-AX):\ F\in\mathbb{C}^{n\times n}\}$.

Proof. (i) The set of all minimal rank weak Drazin inverses of A is given in [3, Theorem 2] as $\{A^D+A^DAE(I-A^DA):\ E\in\mathbb{C}^{n\times n}\}$. Using the definition of weak MPCEP inverse, we compete this part.

(ii) $\wedge$ (iii) The set of all minimal rank weak Drazin inverses of A is presented in [21, Theorem 2.8] as $\{X+XF(I-XA):\ F\in\mathbb{C}^{n\times n}\}$ or $\{X+XF(I-AX):\ F\in\mathbb{C}^{n\times n}\}$. □

4 Perturbation results of weak MPCEP and *CEPMP inverses

One significant part of the theory of generalized inverses is perturbation theory. Perturbation results for the Moore-Penrose inverse are given in [2] and for the minimal rank weak Drazin inverse in [14].

Perturbation expressions and perturbation bounds of weak MPCEP inverse are firstly investigated in this section.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$, X be a minimal rank weak Drazin inverse of A, $Y = A^\dagger AX$ and $B = A + E \in \mathbb{C}^{n\times n}$. If $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$, $\mathcal{R}(E^*) \subseteq \mathcal{R}((XA)^*)$ and $\max\{\|EX\|, \|A^\dagger E\|\} < 1$, then*

$$(I + YE)^{-1}Y = B^\dagger B(I + XE)^{-1}X$$

and

$$Y(I + EY)^{-1} = B^\dagger BX(I + EX)^{-1},$$

where $X(I+EX)^{-1}$ is a minimal rank weak Drazin inverse of B. In addition,

$$B(I + XE)^{-1}X = B(B^\dagger B(I + XE)^{-1}X) = AY,$$

$$B^\dagger BX(I + EX)^{-1}B = YA,$$

$$\frac{\|Y\|}{1 + \|YE\|} \le \|B^\dagger B(I + XE)^{-1}X\| \le \frac{\|Y\|}{1 - \|YE\|}$$

and

$$\frac{\|Y\|}{1 + \|EY\|} \le \|B^\dagger BX(I + EX)^{-1}\| \le \frac{\|Y\|}{1 - \|EY\|}.$$

Proof. By [2], $\mathcal{R}(E) \subseteq \mathcal{R}(A^k) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(E^*) \subseteq \mathcal{R}((XA)^*) \subseteq \mathcal{R}(A^*)$ imply

$$B^\dagger = (I + A^\dagger E)^{-1}A^\dagger = A^\dagger(I + EA^\dagger)^{-1},$$

$$BB^\dagger = AA^\dagger, \qquad B^\dagger B = A^\dagger A.$$

Because $\mathcal{R}(E) \subseteq \mathcal{R}(A^k) = \mathcal{R}(X)$, we deduce that $E = XU = AX^2U = AXE$. From $\mathcal{R}(E^*) \subseteq \mathcal{R}(A^*)$, $E = EA^\dagger A$. The assumption $\|A^\dagger E\| < 1$ gives that $I + YE = I + A^\dagger AXE = I + A^\dagger E$ is nonsingular. Using $\|EX\| < 1$, $I + EY = I + EA^\dagger AX = I + EX$ is nonsingular. According to [14, Theorem 4.1],

$$\begin{aligned}(I + YE)^{-1}Y &= (I + A^\dagger E)^{-1}A^\dagger(AX)\\ &= B^\dagger B(I + XE)^{-1}X\end{aligned}$$

and

$$Y(I+EY)^{-1} \quad = \quad A^{\dagger}AX(I+EX)^{-1} = B^{\dagger}BX(I+EX)^{-1}.$$

Since $B = A+E = A+AXE = A+AA^{\dagger}AXE = A(I+YE)$, we conclude that $B(I+XE)^{-1}X = B(B^{\dagger}B(I+XE)^{-1}X) = A(I+YE)(I+YE)^{-1}Y = AY$. Notice that $\mathcal{R}(E^*) \subseteq \mathcal{R}((XA)^*)$ yields $E = HXA = (HXA)XA = EXA$, for some $H \in \mathbb{C}^{n\times n}$. Therefore, $B = (I+EX)A = (I+EY)A$ and $(B^{\dagger}BX(I+EX)^{-1})B = Y(I+EY)^{-1}(I+EY)A = YA$. □

By Theorem 4.1, we can get as a special case perturbation formulae for the MPD inverse given in [14, Corollary 4.1(i)].

For $X = A^{\oplus}$ in Theorem 4.1, perturbation results for the MPCEP inverse can be established.

Corollary 4.1. *Let $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ and $B = A+E \in \mathbb{C}^{n\times n}$. If $\mathcal{R}(E) \subseteq \mathcal{R}(A^k)$, $\mathcal{R}(E^*) \subseteq \mathcal{R}(A^*A^k)$ and $\max\{\|EA^{\oplus}\|, \|A^{\dagger}E\|\} < 1$, then*

$$B^{\dagger,\oplus} \quad = \quad (I+A^{\dagger,\oplus}E)^{-1}A^{\dagger,\oplus} = A^{\dagger,\oplus}(I+EA^{\dagger,\oplus})^{-1},$$

$BB^{\dagger,\oplus} = AA^{\dagger,\oplus}$, $B^{\dagger,\oplus}B = A^{\dagger,\oplus}A$ and

$$\frac{\|A^{\dagger,\oplus}\|}{1+\|A^{\dagger,\oplus}E\|} \le \|B^{\dagger,\oplus}\| \le \frac{\|A^{\dagger,\oplus}\|}{1-\|A^{\dagger,\oplus}E\|}.$$

Proof. According to [13, Corollary 3.1] and $AA^{\oplus}E = A^k(A^k)^{\dagger}E = E$, $B^{\oplus} = (I+A^{\oplus}E)^{-1}A^{\oplus} = A^{\oplus}(I+EA^{\oplus})^{-1}$. The rest follows by Theorem 4.1. □

Perturbation expressions of the weak *CEPMP inverse can be proved in an analogy manner.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$, Z be a minimal rank right weak Drazin inverse of A, $Y = ZAA^{\dagger}$ and $B = A+E \in \mathbb{C}^{n\times n}$. If $\mathcal{R}(E) \subseteq \mathcal{R}(AZ)$, $\mathcal{R}(E^*) \subseteq \mathcal{R}((A^k)^*)$ and $\max\{\|ZE\|, \|EA^{\dagger}\|\} < 1$, then*

$$(I+YE)^{-1}Y = (I+ZE)^{-1}ZBB^{\dagger}$$

and

$$Y(I+EY)^{-1} = Z(I+EZ)^{-1}BB^{\dagger},$$

where $(I+ZE)^{-1}Z$ is a minimal rank right weak Drazin inverse of B. In addition,

$$B(I+ZE)^{-1}ZBB^{\dagger} = AY,$$

$$Z(I+EZ)^{-1}B = (Z(I+EZ)^{-1}BB^{\dagger})B = YA,$$

$$\frac{\|Y\|}{1+\|YE\|} \leq \|(I+ZE)^{-1}ZBB^{\dagger}\| \leq \frac{\|Y\|}{1-\|YE\|}$$

and

$$\frac{\|Y\|}{1+\|EY\|} \leq \|Z(I+EZ)^{-1}BB^{\dagger}\| \leq \frac{\|Y\|}{1-\|EY\|}.$$

When $Z = A_{\textcircled{\dagger}}$ in Theorem 4.2, we obtain the next perturbation results for the *CEPMP inverse.

Corollary 4.2. *Let* $A \in \mathbb{C}^{n\times n}$*,* $k = \mathrm{ind}(A)$ *and* $B = A + E \in \mathbb{C}^{n\times n}$*. If* $\mathcal{R}(E) \subseteq \mathcal{R}(A(A^k)^*)$*,* $\mathcal{R}(E^*) \subseteq \mathcal{R}((A^k)^*)$ *and* $\max\{\|A_{\textcircled{\dagger}}E\|, \|EA^{\dagger}\|\} < 1$*, then*

$$B_{\textcircled{\dagger},\dagger} \;=\; (I + A_{\textcircled{\dagger},\dagger}E)^{-1}A_{\textcircled{\dagger},\dagger} = A_{\textcircled{\dagger},\dagger}(I + EA_{\textcircled{\dagger},\dagger})^{-1},$$

$BB_{\textcircled{\dagger},\dagger} = AA_{\textcircled{\dagger},\dagger}$*,* $B_{\textcircled{\dagger},\dagger}B = A_{\textcircled{\dagger},\dagger}A$ *and*

$$\frac{\|A_{\textcircled{\dagger},\dagger}\|}{1+\|A_{\textcircled{\dagger},\dagger}E\|} \leq \|B_{\textcircled{\dagger},\dagger}\| \leq \frac{\|A_{\textcircled{\dagger},\dagger}\|}{1-\|A_{\textcircled{\dagger},\dagger}E\|}.$$

5 Applications of weak MPCEP and *CEPMP inverses

Solvability of certain linear equations can be obtained applying weak MPCEP and *CEPMP inverses.

Theorem 5.1. *Let* $B \in \mathbb{C}^{n\times q}$*,* $A \in \mathbb{C}^{n\times n}$*,* $k = \mathrm{ind}(A)$ *and* X *be a minimal rank weak Drazin inverse of* A*. For* $Y = A^{\dagger}AX$*, the equation (with unknown* U*)*

$$XAU = XB \tag{11}$$

has the general solution in the next form

$$U = YB + (I - YA)V, \tag{12}$$

for arbitrary $V \in \mathbb{C}^{n\times q}$*.*

Proof. Theorem 2.2 implies that $XAY = X$. If U has the form (12), we get

$$XAU = XAYB + XA(I - YA)V = XB,$$

that is, (11) holds.

When the equation (11) has a solution U, then

$$YAU = A^{\dagger}A(XAU) = A^{\dagger}AXB = YB.$$

Hence, $U = YB + U - YAU = YB + (I - YA)U$ has the form as in (12). □

Consequently, Theorem 5.1 gives the next result.

Corollary 5.1. *If $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$ and $k = \operatorname{ind}(A)$, the equation*

(i) $A^D AU = A^D B$ *has the general solution* $U = A^{\dagger,D} B + (I - A^{\dagger,D} A)V$;

(ii) $A^{\oplus} AU = A^{\oplus} B$ *has the general solution* $U = A^{\dagger,\oplus} B + (I - A^{\dagger,\oplus} A)V$;

(iii) $A^{\circledS_m} AU = A^{\circledS_m} B$*, where* $m \in \mathbb{N}$*, has the general solution* $U = A^\dagger A A^{\circledS_m} B + (I - A^\dagger A A^{\circledS_m} A)V$.

for arbitrary $V \in \mathbb{C}^{n\times q}$.

As Theorem 5.1, we can verify solvability of the next equations.

Theorem 5.2. *Let $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. For $Y = A^\dagger AX$, the equation*

$$AU = AXB$$

has the general solution in the next form

$$U = YB + (I - A^\dagger A)V,$$

for arbitrary $V \in \mathbb{C}^{n\times q}$.

If we add the condition $\mathcal{R}(B) \subseteq \mathcal{R}(A^k)$ in Theorem 5.2, we obtain the main well-known role of the Moore-Penrose inverse in solving linear equations [12].

Corollary 5.2. *Let $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. For $Y = A^\dagger AX$, the equation*

$$AU = B, \qquad \mathcal{R}(B) \subseteq \mathcal{R}(A^k),$$

has the general solution in the next form

$$U = A^\dagger B + (I - A^\dagger A)V,$$

for arbitrary $V \in \mathbb{C}^{n\times q}$.

Proof. Since $\mathcal{R}(B) \subseteq \mathcal{R}(A^k) = \mathcal{R}(X)$ and $X = AX^2$, we conclude that $B = XH = AX(XH) = AXB$, for some $H \in \mathbb{C}^{n\times q}$. The rest is clear by Theorem 5.2. □

Theorem 5.3. *Let $B \in \mathbb{C}^{q\times n}$, $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ and X be a minimal rank weak Drazin inverse of A. For $Y = A^\dagger AX$, the equation*

$$UA^k = BA^\dagger A^k$$

has the general solution in the next form

$$U = BY + V(I - AX),$$

for arbitrary $V \in \mathbb{C}^{q\times n}$.

Remark that the equation $UA^k = BA^\dagger A^k$ is equivalent to $UA^D = BA^\dagger A^D$ and so these equalities have the solution in the same form.

Analogously, we solve several linear equations using weak *CEPMP inverse.

Theorem 5.4. *Let* $B \in \mathbb{C}^{q\times n}$, $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ *and* Z *be a minimal rank right weak Drazin inverse of* A. *For* $Y = ZAA^\dagger$, *the equation*

$$UAZ = BZ$$

has the general solution in the next form

$$U = BY + V(I - AY),$$

for arbitrary $V \in \mathbb{C}^{q\times n}$.

Corollary 5.3. *If* $B \in \mathbb{C}^{q\times n}$, $A \in \mathbb{C}^{n\times n}$ *and* $k = \mathrm{ind}(A)$, *the equation*

(i) $UA^DA = BA^D$ *has the general solution* $U = BA^{D,\dagger} + V(I - AA^{D,\dagger})$;

(ii) $UAA_{\oplus} = BA_{\oplus}$ *has the general solution* $U = BA_{\oplus,\dagger} + V(I - AA_{\oplus,\dagger})$.

for arbitrary $V \in \mathbb{C}^{q\times n}$.

Theorem 5.5. *Let* $B \in \mathbb{C}^{q\times n}$, $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ *and* Z *be a minimal rank right weak Drazin inverse of* A. *For* $Y = ZAA^\dagger$, *the equation*

$$UA = BZA$$

has the general solution in the next form

$$U = BY + V(I - AA^\dagger),$$

for arbitrary $V \in \mathbb{C}^{q\times n}$.

Corollary 5.4. *Let* $B \in \mathbb{C}^{q\times n}$, $A \in \mathbb{C}^{n\times n}$, $k = \mathrm{ind}(A)$ *and* Z *be a minimal rank right weak Drazin inverse of* A. *For* $Y = ZAA^\dagger$, *the equation*

$$UA = B, \qquad \mathcal{N}(A^k) \subseteq \mathcal{N}(B),$$

has the general solution in the next form

$$U = BA^\dagger + V(I - AA^\dagger),$$

for arbitrary $V \in \mathbb{C}^{q\times n}$.

Theorem 5.6. *Let $B \in \mathbb{C}^{n\times q}$, $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$ and Z be a minimal rank right weak Drazin inverse of A. For $Y = ZAA^\dagger$, the equation*

$$A^k U = A^k A^\dagger B$$

has the general solution in the next form

$$U = YB + (I - ZA)V,$$

for arbitrary $V \in \mathbb{C}^{n\times q}$.

Theorem 5.1 can be confirmed in the following example.

Example 5.1. *Let A be the matrix as in Example 2.1,*

$$B = \begin{bmatrix} 10 & 20 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \\ v_5 & v_6 \end{bmatrix}.$$

Since

$$\begin{aligned} U &= YB + (I - YA)V \\ &= \begin{bmatrix} 2 + \frac{1}{5}v_1 - \frac{2}{5}v_3 - \frac{4}{5}x_1v_5 & 4 + \frac{1}{5}v_2 - \frac{2}{5}v_4 - \frac{4}{5}x_1v_6 \\ 1 - \frac{2}{5}v_1 + \frac{4}{5}v_3 - \frac{2}{5}x_1v_5 & 2 - \frac{2}{5}v_2 + \frac{4}{5}v_4 - \frac{2}{5}x_1v_6 \\ v_5 & v_6 \end{bmatrix}, \end{aligned}$$

we verify that

$$XAU = \begin{bmatrix} \frac{5}{2} & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = XB,$$

i.e. Theorem 5.1 is confirmed.

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